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Any l -state solutions of the Hulthén potential by the asymptotic iteration method

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Abstract

In this paper, we present the analytical solution of the radial Schrödinger equation for the Hulthén potential within the framework of the asymptotic iteration method by using an approximation to the centrifugal potential for any l states. We obtain the energy eigenvalues and corresponding eigenfunctions for different screening parameters. The wavefunctions are physical and energy eigenvalues are in good agreement with the results obtained by other methods for different δ values. In order to demonstrate this, the results of the asymptotic iteration method are compared with the results of the supersymmetry, numerical integration, variational and shifted $1/N$ expansion methods.

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1. Introduction

Over the last few decades, the energy eigenvalues and corresponding eigenfunctions between interaction systems have raised a great deal of interest in relativistic quantum mechanics as well as in non-relativistic quantum mechanics. The exact solution of the wave equations (relativistic or non-relativistic) is very important since the wavefunction contains all the necessary information regarding the quantum system under consideration. Analytical methods such as the supersymmetry (SUSY) [1] and Nikiforov–Uvarov (NU) methods [2] have been used to solve the wave equations exactly or quasi-exactly for non-zero angular momentum quantum number ($l \neq 0$) by means of a given potential. The radial Schrödinger equation for the Hulthén potential is solved exactly by using several techniques [6–8] for $l = 0$. For the case $l \neq 0$, the effective Hulthén potential cannot be solved exactly, but a number of methods have been used to find the bound-state energy eigenvalues numerically [9] and quasi-analytically, such as the variational [9], perturbation [10], shifted $1/N$ expansion [11], NU [12] and SUSY [13] methods. The Hulthén potential [6] is one of the important short-range potentials in physics and it has been applied to a number of areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics (see [9] and references

therein). Therefore, it would be interesting and important to solve the non-relativistic radial Schrödinger equation for this potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interaction systems. Recently, an alternative method, called the asymptotic iteration method (AIM), has been developed by Çiftçi *et al* [3, 4] for solving second-order homogeneous linear differential equations and it has been applied to solve the non-relativistic radial Schrödinger equation or the relativistic Dirac equation.

In this paper, we aim to solve the Hulthén potential to obtain the energy eigenvalues and corresponding eigenfunctions for any l states. In the next section, AIM is introduced. Then, in section 3, the Schrödinger equation is solved by using AIM for the Hulthén potential for any l states and our AIM results are given in comparison with the results of the numerical integration [9], variational [9], shifted $1/N$ expansion [11] and SUSY [13] methods. Finally, section 4 is devoted to the summary and conclusion.

2. Overview of the asymptotic iteration method

2.1. Energy eigenvalues

AIM is briefly outlined here and the details can be found in [3–5]. AIM is proposed to solve the second-order differential equations of the form

$$y'' = \lambda_0(x)y' + s_0(x)y, \quad (1)$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x . The variables $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. The differential equation (1) has a general solution [3]

$$y(x) = \exp\left(-\int^x \alpha(x_1) dx_1\right) \left[C_2 + C_1 \int^x \exp\left(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)] dx_2\right) dx_1 \right] \quad (2)$$

for sufficiently large k , $k > 0$, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \quad k = 1, 2, 3, \dots, \quad (3)$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x), \quad k = 1, 2, 3, \dots \end{aligned} \quad (4)$$

Note that one can also start the recurrence relations from $k = 0$ with the initial conditions $\lambda_{-1} = 1$ and $s_{-1} = 0$ [20]. For a given potential such as the Hulthén potential, the radial Schrödinger equation is converted to the form of equation (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by equation (4).

The termination condition of the method in equation (3) can be arranged as

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \quad k = 1, 2, 3, \dots \quad (5)$$

The energy eigenvalues are obtained from the roots of equation (5) if the problem is exactly solvable. If not, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wavefunction or the minimum value of the potential [3, 18–20], and the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently large values of k with iteration.

2.2. Energy eigenfunctions

In this study, we seek the exact solution of the radial Schrödinger equation for which the relevant second-order homogenous linear differential equation takes the following general form [4]:

$$y'' = 2 \left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(t+1)}{x} \right) y' - \frac{w_k^t(N)x^N}{1-bx^{N+2}} y, \quad 0 < x < \infty. \quad (6)$$

If this equation is compared to equation (1), it entails the following expressions:

$$\lambda_0(x) = 2 \left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(t+1)}{x} \right), \quad s_0(x) = -\frac{w_k^t(N)x^N}{1-bx^{N+2}}. \quad (7)$$

a and b are constants and $w_k^t(N)$ can be determined from condition (3) for $k = 0, 1, 2, 3, \dots$ and $N = -1, 0, 1, 2, 3, \dots$ as follows:

$$w_k^t(-1) = k(2a + 2bt + (k+1)b), \quad (8)$$

$$w_k^t(0) = 2k(2a + 2bt + (2k+1)b), \quad (9)$$

$$w_k^t(1) = 3k(2a + 2bt + (3k+1)b), \quad (10)$$

$$w_k^t(2) = 4k(2a + 2bt + (4k+1)b), \quad (11)$$

$$w_k^t(3) = 5k(2a + 2bt + (5k+1)b), \quad (12)$$

...

Hence, these formulae are easily generalized as

$$w_k^t(N) = b(N+2)^2 k \left(k + \frac{(2t+1)b+2a}{(N+2)b} \right). \quad (13)$$

The exact eigenfunctions can be derived from the following generator:

$$y_n(x) = C_2 \exp \left(- \int^x \frac{s_k(x')}{\lambda_k(x')} dx' \right), \quad (14)$$

where $k \geq n$, n represents the radial quantum number and k denotes the iteration number. For exactly solvable potentials, the radial quantum number n is equal to the iteration number k and the eigenfunctions are obtained directly from equation (14). For nontrivial potentials that have no exact solutions, k is always larger than n in these numerical solutions and the approximate energy eigenvalues are obtained from the roots of equation (5) for sufficiently large values of k with iteration. It should be pointed out that $\alpha(x)$ given by equation (3) is equal to zero for the ground state. Therefore, using equation (3) with (7) in equation (14), the eigenfunctions are obtained as follows:

$$\begin{aligned} y_0(x) &= C_2, & y_1(x) &= -C_2(N+2)\sigma \left(1 - \frac{b(\rho+1)}{\sigma} x^{N+2} \right), \\ y_2(x) &= C_2(N+2)^2 \sigma(\sigma+1) \left(1 - \frac{2b(\rho+2)}{\sigma} x^{N+2} + \frac{b^2(\rho+2)(\rho+3)}{\sigma(\sigma+1)} x^{2(N+2)} \right), \\ y_3(x) &= -C_2 \frac{\sigma(\sigma+1)(\sigma+2)}{(N+2)^3} \left(1 - \frac{3b(\rho+3)}{\sigma} x^{N+2} + \frac{3b^2(\rho+3)(\rho+4)}{\sigma(\sigma+1)} x^{2(N+2)} \right. \\ &\quad \left. - \frac{b^3(\rho+3)(\rho+4)(\rho+5)}{\sigma(\sigma+1)(\sigma+2)} x^{3(N+2)} \right), \\ &\dots \end{aligned}$$

Finally, the following general formula for the exact solutions $y_n(x)$ is obtained as

$$y_n(x) = (-1)^n C_2(N+2)^n (\sigma)_n {}_2F_1(-n, \rho+n; \sigma; bx^{N+2}). \quad (15)$$

It is important to note that the square-integrable one in L^2 is this total wavefunction which is the asymptotic form of the wavefunction times $y_n(x)$ given by equation (14). Here, $(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}$, $\sigma = \frac{2l+N+3}{N+2}$ and $\rho = \frac{(2l+1)b+2a}{(N+2)b}$. $(\sigma)_n$ and ${}_2F_1$ are known as the Pochhammer symbol and the Gauss hypergeometric function, respectively.

3. Calculation of the energy eigenvalues and eigenfunctions

The motion of a particle with the mass M in the spherically symmetric potential is described in the spherical coordinates by the following Schrödinger equation:

$$\frac{-\hbar^2}{2M} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + V(r) \right) \Psi_{nlm}(r, \theta, \phi) = E \Psi_{nlm}(r, \theta, \phi). \quad (16)$$

Defining $\Psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$, we obtain the radial part of the Schrödinger equation:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R_{nl}(r) + \frac{2M}{\hbar^2} \left[E - V(r) - \frac{l(l+1)\hbar^2}{2Mr^2} \right] R_{nl}(r) = 0. \quad (17)$$

It is sometimes convenient to define $R_{nl}(r)$ and the effective potential as follows:

$$R_{nl}(r) = \frac{u_{nl}(r)}{r}, \quad V_{\text{eff}} = V(r) + \frac{l(l+1)\hbar^2}{2Mr^2}. \quad (18)$$

Since

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \frac{u_{nl}(r)}{r} = \frac{1}{r} \frac{d^2}{dr^2} u_{nl}(r), \quad (19)$$

the radial Schrödinger equation [8] given by equation (17) follows that

$$\frac{d^2 u_{nl}(r)}{dr^2} + \frac{2M}{\hbar^2} [E - V_{\text{eff}}] u_{nl}(r) = 0. \quad (20)$$

The Hulthén potential [6] is given by

$$V_H(r) = -Ze^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \quad (21)$$

where Z and δ are, respectively, the atomic number and the screening parameter, determining the range for the Hulthén potential. The Hulthén potential behaves like the Coulomb potential near the origin ($r \rightarrow 0$), but in the asymptotic region ($r \gg 1$) the Hulthén potential decreases exponentially, so its capacity for bound states is smaller than the Coulomb potential. However, for small values of the screening parameter δ , the Hulthén potential becomes the Coulomb potential given by $V_C = -\frac{Ze^2}{r}$. The effective Hulthén potential is

$$V_{\text{eff}}(r) = V_H(r) + V_l = -Ze^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l+1)\hbar^2}{2Mr^2}, \quad (22)$$

where $V_l = \frac{l(l+1)\hbar^2}{2Mr^2}$ is known as the centrifugal term. This effective potential cannot be solved analytically for $l \neq 0$ because of the centrifugal term. Therefore, we must use an approximation for the centrifugal term similar to other authors [12–17]. In this approximation, $\frac{1}{r^2} = \delta^2 \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2}$ is used for the centrifugal term. As shown in figure 1, this approximation is

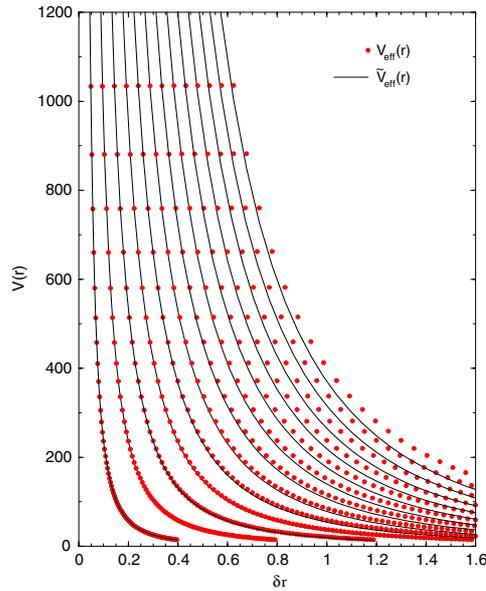


Figure 1. The variation of the effective Hulthén $V_{\text{eff}}(r)$ and approximate effective Hulthén potentials $\tilde{V}_{\text{eff}}(r)$ with respect to δr for various values of the screening parameter. The parameters are in atomic units ($\hbar = e = m = 1$) and δ change from 0.4 to 5.2 in steps of 0.4.

only valid for small δr and it breaks down in the high-screening region. For small δr , $\tilde{V}_{\text{eff}}(r)$ is very well approximated to $V_{\text{eff}}(r)$ and the Schrödinger equation for this approximate potential is solvable analytically. So, the effective potential becomes

$$\tilde{V}_{\text{eff}}(r) = -Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l+1)\hbar^2\delta^2}{2M} \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2}. \quad (23)$$

Instead of solving the radial Schrödinger equation for the effective Hulthén potential $V_{\text{eff}}(r)$ given by equation (22), we now solve the radial Schrödinger equation for the new effective potential $\tilde{V}_{\text{eff}}(r)$ given by equation (23). Inserting this new effective potential into equation (20) and using the following ansätze in order to make the differential equation more compact:

$$-\varepsilon^2 = \frac{2ME}{\hbar^2\delta^2}, \quad \beta^2 = \frac{2MZe^2}{\hbar^2\delta}, \quad \delta r = x, \quad (24)$$

the radial Schrödinger equation takes the following form:

$$\frac{d^2 u_{nl}(x)}{dx^2} + \left[-\varepsilon^2 + \beta^2 \frac{e^{-x}}{(1 - e^{-x})} - l(l+1) \frac{e^{-x}}{(1 - e^{-x})^2} \right] u_{nl}(x) = 0. \quad (25)$$

If we rewrite equation (25) by using a new variable of the form $z = e^{-x}$, we obtain

$$\frac{d^2 u_{nl}(z)}{dz^2} + \frac{1}{z} \frac{du_{nl}(z)}{dz} + \left[-\frac{\varepsilon^2}{z^2} + \frac{\beta^2}{z(1-z)} - \frac{l(l+1)}{z(1-z)^2} \right] u_{nl}(z) = 0. \quad (26)$$

In order to solve this equation with AIM, we should transform this equation to the form of equation (1). Therefore, the reasonable physical wavefunction we propose is as follows:

$$u_{nl}(z) = z^\varepsilon (1-z)^{l+1} f_{nl}(z). \quad (27)$$

If we insert this wavefunction into equation (26), we have the second-order homogeneous linear differential equations in the following form:

$$\frac{d^2 f_{nl}(z)}{dz^2} = \left[\frac{(2\varepsilon + 2l + 3)z - (2\varepsilon + 1)}{z(1-z)} \right] \frac{df_{nl}(z)}{dz} + \left[\frac{(2\varepsilon + l + 2)l + 2\varepsilon - \beta^2 + 1}{z(1-z)} \right] f_{nl}(z), \quad (28)$$

which is now amenable to an AIM solution. By comparing this equation with equation (1), we can write the $\lambda_0(z)$ and $s_0(z)$ values and by means of equation (4) we may calculate $\lambda_k(z)$ and $s_k(z)$. This gives

$$\begin{aligned} \lambda_0(z) &= \left(\frac{(2\varepsilon + 2l + 3)z - (2\varepsilon + 1)}{z(1-z)} \right), \\ s_0(z) &= \left(\frac{(2\varepsilon + l + 2)l + 2\varepsilon - \beta^2 + 1}{z(1-z)} \right), \\ \lambda_1(z) &= \frac{2 + 6\varepsilon - 7z - 2lz - \beta^2 z + 12z^2 l - 18\varepsilon z - 6\varepsilon z l}{z^2 (-1 + z)^2} \\ &\quad + \frac{12\varepsilon z^2 + 11z^2 + 4\varepsilon^2 + l^2 z + \beta^2 z^2 + 4\varepsilon^2 z^2 - 8\varepsilon^2 z + 6\varepsilon z^2 l + 3l^2 z^2}{z^2 (-1 + z)^2}, \\ s_1(z) &= \frac{(2l + 2\varepsilon - \beta^2 + 2\varepsilon l + l^2 + 1)(-2 + 5z + 2\varepsilon z + 2lz - 2\varepsilon)}{z^2 (-1 + z)^2}, \\ &\dots \end{aligned} \quad (29)$$

Combining these results with the quantization condition given by equation (5) yields

$$\begin{aligned} s_0 \lambda_1 - s_1 \lambda_0 = 0 &\Rightarrow \varepsilon_0 = \frac{\beta^2 - 1 - 2l - l^2}{2(l+1)}, \quad \text{for } k = 1, \\ s_1 \lambda_2 - s_2 \lambda_1 = 0 &\Rightarrow \varepsilon_1 = \frac{\beta^2 - 4 - 4l - l^2}{2(l+2)}, \quad \text{for } k = 2, \\ s_2 \lambda_3 - s_3 \lambda_2 = 0 &\Rightarrow \varepsilon_2 = \frac{\beta^2 - 9 - 6l - l^2}{2(l+3)}, \quad \text{for } k = 3, \\ &\dots \end{aligned} \quad (30)$$

When the above expressions are generalized, the eigenvalues turn out to be

$$\varepsilon_{nl} = \left(\frac{\beta^2 - (n+l+1)^2}{2(n+l+1)} \right), \quad n, l = 0, 1, 2, 3, \dots \quad (31)$$

Using equation (24), we obtain the energy eigenvalues E_{nl} ,

$$E_{nl} = -\frac{\hbar^2}{2M} \left[\frac{MZ e^2}{\hbar^2(n+l+1)} - \frac{(n+l+1)\delta}{2} \right]^2. \quad (32)$$

In the atomic units ($\hbar = M = e = 1$) and for $Z = 1$, equation (32) turns out to be

$$E_{nl} = -\frac{1}{2} \left[\frac{1}{(n+l+1)} - \frac{(n+l+1)\delta}{2} \right]^2. \quad (33)$$

In order to test the accuracy of equation (33), we calculate the energy eigenvalues for $Z = 1$, any n and l quantum numbers and several values of the screening parameter. AIM results are compared with the results of the numerical integration [9], variational [9], shifted $1/N$ expansion [11] and SUSY [13] methods in tables 1 and 2. As can be seen from the results presented in these tables, the AIM results are in good agreement with the results of

Table 1. Energy eigenvalues of the Hulthén potential as a function of the screening parameter for 2p, 3p and 3d states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$.

State	δ	AIM	SUSY [13]	Numerical integration [9]	Variational [9]	Shifted $\frac{1}{N}$ [11]
2p	0.025	0.112 8125	0.112 7605	0.112 7605	0.112 7605	
	0.050	0.101 2500	0.101 0425	0.101 0425	0.101 0425	0.101 0424
	0.075	0.090 3125	0.089 8478	0.089 8478	0.089 8478	
	0.100	0.080 0000	0.079 1794	0.079 1794	0.079 1794	0.079 1794
	0.150	0.061 2500	0.059 4415	0.059 4415	0.059 4415	
	0.200	0.045 0000	0.041 8854	0.041 8860	0.041 8860	0.041 8857
	0.250	0.031 2500	0.026 6060	0.026 6111	0.026 6108	
	0.300	0.020 0000	0.013 7596	0.013 7900	0.013 7878	
	0.350	0.011 2500	0.003 6146	0.003 7931	0.003 7734	
3p	0.025	0.043 7590	0.043 7068	0.043 7069	0.043 7069	
	0.050	0.033 3681	0.033 1632	0.033 1645	0.033 1645	0.033 165 18
	0.075	0.024 3837	0.023 9331	0.023 9397	0.023 9397	
	0.100	0.016 8056	0.016 0326	0.016 0537	0.016 0537	0.016 067 72
	0.150	0.005 8681	0.004 3599	0.004 4663	0.004 4660	
3d	0.025	0.043 7587	0.043 6030	0.043 6030	0.043 6030	
	0.050	0.033 3681	0.032 7532	0.032 7532	0.032 7532	0.032 7532
	0.075	0.024 3837	0.023 0306	0.023 0307	0.023 0307	
	0.100	0.016 8055	0.014 4832	0.014 4842	0.014 4842	0.014 4842
	0.150	0.005 8681	0.013 2820	0.001 3966	0.001 3894	

Table 2. Energy eigenvalues of the Hulthén potential as a function of the screening parameter for 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$.

State	δ	AIM	SUSY [13]	Numerical integration [9]	Variational [9]	Shifted $\frac{1}{N}$ [11]
4p	0.025	0.020 0000	0.019 9480	0.019 9489	0.019 9489	
	0.050	0.011 2500	0.011 0430	0.011 0582	0.011 0582	0.011 0725
	0.075	0.005 0000	0.004 5385	0.004 6219	0.004 6219	
	0.100	0.001 2500	0.000 4434	0.000 7550	0.000 7532	
4d	0.025	0.020 0000	0.019 8460	0.019 8462	0.019 8462	
	0.050	0.011 2500	0.010 6609	0.010 6674	0.010 6674	0.010 6690
	0.075	0.005 0000	0.003 7916	0.003 8345	0.003 8344	
4f	0.025	0.020 0000	0.019 6911	0.019 6911	0.019 6911	
	0.050	0.011 2500	0.010 0618	0.010 0620	0.010 0620	0.010 0620
	0.075	0.005 0000	0.002 5468	0.002 5563	0.002 5557	
5p	0.025	0.009 4531	0.009 4011	0.009 4036		0.009 4087
	0.050	0.002 8125	0.002 6056	0.002 6490		
5d	0.025	0.009 4531	0.009 2977	0.009 3037		0.009 3050
	0.050	0.002 8125	0.002 2044	0.002 3131		
5f	0.025	0.009 4531	0.009 1507	0.009 1521		0.009 1523
	0.050	0.002 8125	0.001 7421	0.001 7835		
5g	0.025	0.009 4531	0.008 9465	0.008 9465		0.008 9465
	0.050	0.002 8125	0.001 0664	0.001 0159		
6p	0.025	0.004 2014	0.004 1493	0.004 1548		
6d	0.025	0.004 2014	0.004 0452	0.004 0606		
6f	0.025	0.004 2014	0.003 8901	0.003 9168		
6g	0.025	0.004 2014	0.003 6943	0.003 7201		

the other methods for small δ values. For large δ values, there are differences between our results and the results of others. This difference is due to the $\tilde{V}_{\text{eff}}(r)$ potential, which we have used to approximate the $V_{\text{eff}}(r)$ potential. As seen from figure 1, for large δr values, the discrepancy becomes apparent between our $\tilde{V}_{\text{eff}}(r)$ and true $V_{\text{eff}}(r)$ potentials. This gives rise to the differences for the eigenvalues presented in tables 1 and 2 at large δ values.

Now, as indicated in section 2, we can determine the corresponding wavefunctions by using equation (15). When we compare equations (6) and (28), we find $N = -1$, $b = 1$, $a = l + 1$ and $t = \frac{2\varepsilon - 1}{2}$. Therefore, we find $\rho = 2(\varepsilon + l + 1)$ and $\sigma = 2\varepsilon + 1$. So, we can easily find the solution for $f_{nl}(z)$, for the energy eigenvalue equation (32) by using equation (15):

$$f_{nl}(z) = (-1)^n \frac{\Gamma(2\varepsilon_n + n + 1)}{\Gamma(2\varepsilon_n + 1)} {}_2F_1(-n, 2\varepsilon_n + 2l + 2 + n; 2\varepsilon_n + 1; z). \quad (34)$$

Thus, we can write the total radial wavefunction as follows:

$$u_{nl}(z) = Nz^{\varepsilon_n} (1 - z)^{l+1} {}_2F_1(-n, 2(\varepsilon_n + l + 1) + n; 2\varepsilon_n + 1; z), \quad (35)$$

where N is the normalization constant.

4. Conclusion

We have shown an alternative method to obtain the energy eigenvalues and corresponding eigenfunctions of the Hulthén potential within the framework of the asymptotic iteration method for any l states. We have calculated the energy eigenvalues for the Hulthén potential with $Z = 1$ and several values of the screening parameter. The wavefunctions are physical and energy eigenvalues are in good agreement with the results obtained by other methods. In order to demonstrate this, AIM results have been compared with the results of the numerical integration [9], variational [9], shifted $1/N$ expansion [11] and SUSY [13] methods in tables 1 and 2. For small δ values, AIM results are in good agreement with the results of the other methods, but in the high-screening region the agreement is poor. The reason is simply that when δr increases in the high-screening region, the agreement between $V_{\text{eff}}(r)$ and $\tilde{V}_{\text{eff}}(r)$ potentials decreases as shown in figure 1. This problem could be solved by making a better approximation of the centrifugal term.

It should be pointed out that the asymptotic iteration method gives the eigenvalues directly by transforming the radial Schrödinger equation into a form of $y'' = \lambda_0(r)y' + s_0(r)y$. The wavefunctions are easily constructed by iterating the values of $s_0(r)$ and $\lambda_0(r)$. The asymptotic iteration method results in exact analytical solutions if there is any and provides the closed forms for the energy eigenvalues as well as the corresponding eigenfunctions. Where there is no such solution, the energy eigenvalues are obtained by using an iterative approach [18–20]. As presented, AIM puts no constraint on the potential parameter values involved and is easy to implement. The results are sufficiently accurate for practical purposes.

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